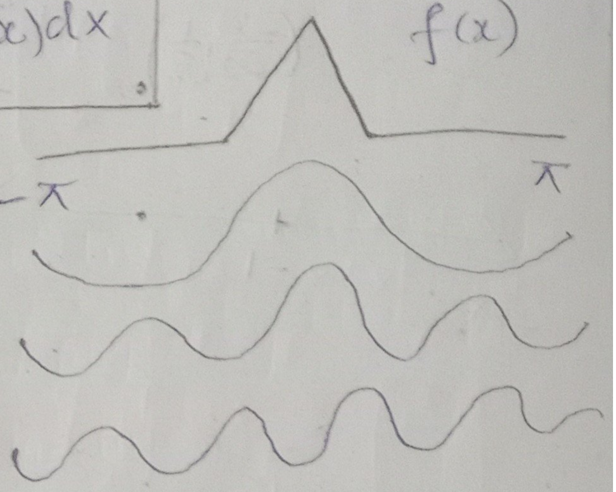
Fourier Series:-

It tells that any arbitary function can be approximated as infinite sum of sines and cosines of increasingly high frequency.

$$\boxed{\langle f(x), g(x) \rangle = \int_a^b f(x)\bar{g}(x)\,dx}$$ → f(x), g(x) defined in [a,b]



These are 2π periodic sine and cosines since f(x) is 2π periodic

$$f(x) = \frac{a_0}{2} + \sum_{K=1}^{\infty}\left(a_K\cos(Kx) + b_K\sin(Kx)\right)$$

$$a_K = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(Kx)\,dx = \frac{\langle f(x), \cos(Kx)\rangle}{\|\cos(Kx)\|^2}$$

$$b_K = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(Kx)\,dx = \frac{\langle f(x), \sin(Kx)\rangle}{\|\sin(Kx)\|^2}$$

Notation:

$\langle a, b\rangle$ — Inner product of $\ddot{a}$ and $\ddot{b}$

norm or length of a $\Rightarrow \langle a, a\rangle = \|a\|^2$

Note: Dot product is one of the form of inner product.

Draw the fourier:

1. Inner product: In essence, an inner product is a function that takes two vectors from vector space and gives you a scalar (a number).

* It's a generalization of the familiar dot product in Euclidean spaces.

Typical Notation: $\langle u, v \rangle$ where $u, v \in V$

V - vector space.

2. Why Inner product spaces Matter:

It enables us to define geometric concepts like:

- Length (Norm): The magnitude of a vector can be found using inner product:

$$||u|| = \sqrt{\langle u, u \rangle}$$

- Angles:

$$\cos\theta = \frac{(\langle u, v \rangle)}{||u|| \, ||v||}$$

** Two vectors are orthogonal if their inner product is zero

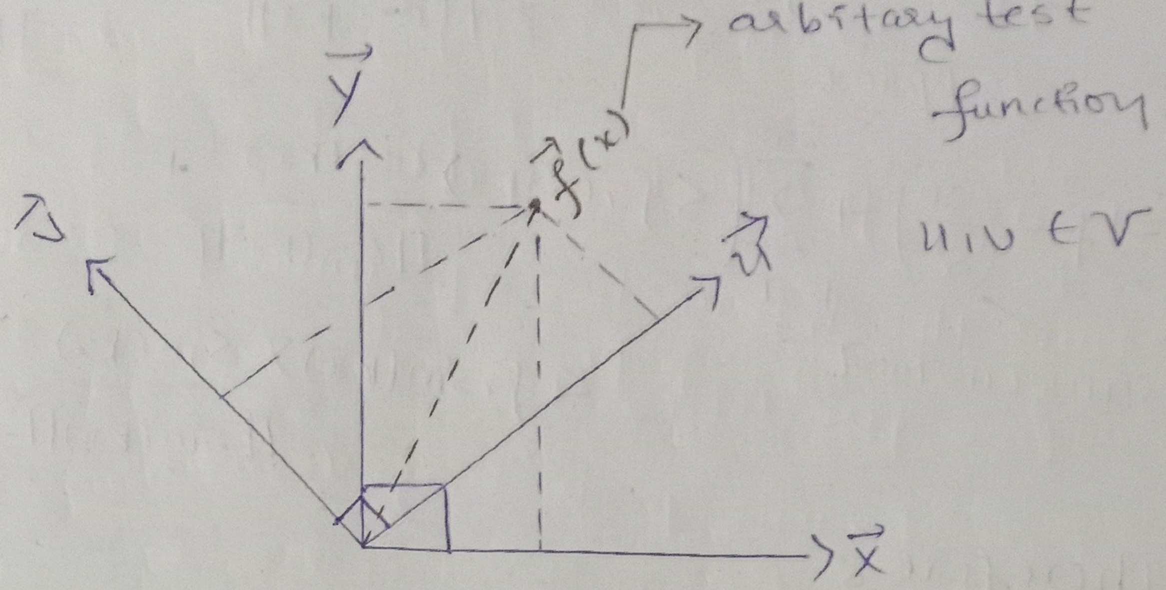
** The Idea of projecting one vector onto another has numerous applications and makes use of the inner product. This leads to concepts like finding closest approximation of a function within a subspace.

** Least Squares Approximation: We find the "best fit" line minimizing the error by orthogonally projecting data points onto subspace defining the line.

** Gram-Schmidt process:- This powerful algorithm allows us to create an orthonormal basis (a set of unit length and mutually orthogonal vectors) from any basis within an inner product space

** Hilbert Spaces:- These are special type of inner product spaces that are complete (no gaps or 'missing points). They provide the foundation for quantum mechanics, $\boxed{\text{Fourier analysis}}$ and much of functional analysis.

Now lets draw fourier:



$\bar{x}\bar{y}$ - coordinate system (orthogonal basis) of R^2

$\bar{u}\bar{v}$ - " "

* If we want to represent $\vec{f}$ in xy system. we take the projection of $\vec{f}$ along $\vec{x}$ direction (It is given by inner product of f, x) and along $\bar{y}$ direction and multiply with respective unit vectors!!

$$\vec{f} = \langle \bar{f}, \bar{x} \rangle \frac{\bar{x}}{\|x\|^2} + \langle \bar{f}, \bar{y} \rangle \frac{\bar{y}}{\|\bar{y}\|^2}$$

$$\vec{f} \approx \langle \bar{f}, \bar{u} \rangle \frac{\vec{u}}{\|\bar{u}\|^2} + \langle \bar{f}, \bar{v} \rangle \frac{\bar{v}}{\|\bar{v}\|^2}$$

<u>Conclusion:</u>

If $\bar{u} = \cos(Kx)$ and $\bar{v} = \sin(Kx)$

$$\bar{f} \approx \langle \bar{f}, \cos(kx) \rangle \frac{\cos(kx)}{\|\cos kx\|^2} + \frac{\langle \bar{f}, \sin kx \rangle \sin}{\|\sin kx\|}$$

approx

$$\Rightarrow \vec{f} = \left(\frac{a_0}{2}\right) + \sum_{k=1}^{\infty} \left(\langle \bar{f}, \cos kx \rangle \frac{\cos(kx)}{\|\cos(kx)\|^2} + \langle \bar{f}, \sin(kx) \rangle \frac{\sin(kx)}{\|\sin(kx)\|^2} \right)$$

some constant

Therefore :-

fourier series is nothing but projecting any arbitary function ($\bar{f}$) onto orthogonal basis $\cos kx$ and $\sin(kx)$.

* Before I wrote:

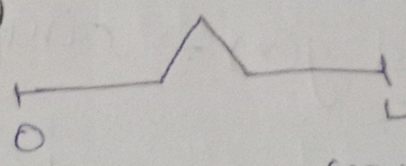
$$a_k = \langle \bar{f}, \cos(kx) \rangle \frac{1}{\|\cos(kx)\|^2}$$

= It is nothing but how much of my function $\bar{f}$ is along $\cos(kx)$ function direction.

$\|^{ly}$ b_k "

* If our $f(x)$ defined from 0 to L

Then: $f(x) \in L_2([0,L])$



$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{2\pi k}{L}x\right) + b_k \sin\left(\frac{2\pi k}{L}x\right)\right)$$

Reason: we want to make sine and cosine periodic between 0 to L ($\because$ our f is also periodic from 0 to L).

$$a_k = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi k}{L}x\right)dx$$

$$b_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi k}{L}x\right)dx$$

Remember / Recall:- $x \in [a,b]$

$$\langle f(x), g(x)\rangle = \int_a^b f(x)\, g(x)\, dx$$

$\Downarrow$

Inner product

is defined like this if $f(x)$ and $g(x)$ are real valued functions

$$\langle f(x), g(x)\rangle = \int_a^b f(x)\, \bar{g}(x)\, dx$$

if $f(x)$ and $g(x)$ are complex valued function, and $\overline{g}(x) \to$ complex conjugate.

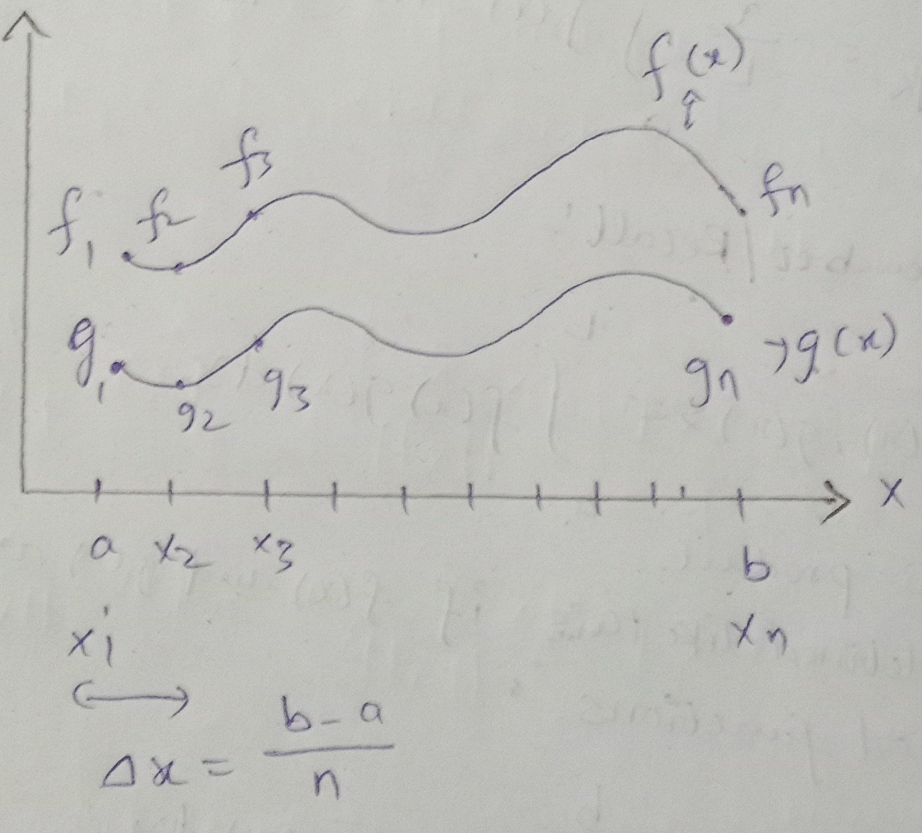
Inner products in Hilbert space :-

<u>Aim</u>:- The definition of inner product of two functions tells exact same information of inner product of vectors.

$$\langle f(x), g(x) \rangle = \int_a^b f(x)\, g(x)\, dx$$

$\downarrow$

Inner product definition of two functions. Tells how functions are alligned in functional space



$$\Delta x = \frac{b-a}{n}$$

$$\underline{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \qquad \underline{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$$

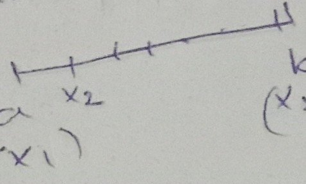
$\downarrow$

functional vector

$$\langle \underline{f}\ \underline{g} \rangle = \underline{g}^T \underline{f} = \begin{bmatrix} \quad g \quad \end{bmatrix} \begin{bmatrix} f \end{bmatrix} = \sum_{k=1}^{n} f_k g_k$$

observe!:

If we sample $[a, b]$ to $2n$ points



$$\text{then } \langle \underline{f}\ \underline{g} \rangle = \sum_{k=1}^{2n} f_k g_k = 2\sum_{k=1}^{n} f_k g_k$$

what is the issue here?

we are trying to normalize it with Δx

$$\boxed{\langle \underline{f}\ \underline{g} \rangle \Delta x = \sum_{k=1}^{n} f(x_k)\, g(x_k)\, \Delta x}$$

$\hookrightarrow$ It is just the reman approximation. I

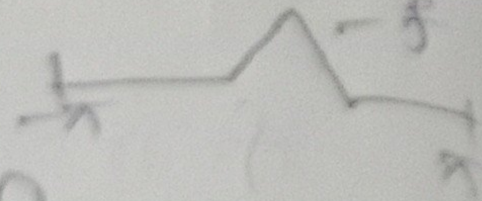
will of $\langle f(x), g(x) \rangle$ when we make Δx

be equal to $\int_a^b f(x)\, g(x)\, dx$

infinitely small.

** We have complex fourier series as well.

* Lets try to show sin(kx) and cos(kx) are orthogonal basis in $[-\pi, \pi]$



$\Rightarrow$ To show $\Rightarrow$ $\langle \sin(kx), \cos(kx) \rangle = 0$

we know

$$\langle f(x), g(x) \rangle = \int_a^b f(x)\, g(x)\, dx$$

$$\Rightarrow \langle \sin kx, \cos kx \rangle = \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} \sin(kx) \cos(kx)\, dx$$

(odd) $\times$ (even) = odd

$$= 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n \cos nx\,dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} \underbrace{b_n \sin nx}_{\text{odd } f^n}\,dx$$

$$= \frac{a_0}{2}(2\pi) + \sum_{n=1}^{\infty} a_n \left(\frac{\sin nx}{n}\right)_{-\pi}^{\pi} + 0$$

$$\left(\because \int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & f(x) \text{ is odd } f^n \\ 2\int_0^a f(x)\,dx, & f(x) \text{ is even } f^n \end{cases}\right.$$

Pf:-

$$\int_{-a}^{a} f(x)\,dx = \underbrace{\int_{-a}^{0} f(x)\,dx}_{I} + \int_0^a f(x)\,dx$$

$$I = \int_{-a}^{0} f(x)\,dx$$

$x = -y$

$$= -\int_a^0 f(-y)\,dy$$

$$= \begin{cases} \int_0^a f(y)\,dy, & \text{if } f \text{ is even } f^n \\ -\int_0^a f(y)\,dy, & \text{if } f \text{ is } \cancel{\text{even}} \text{ odd } f^n \end{cases}$$

$$= \begin{cases} \int_0^a f(x)\,dx, & f \to \text{even} \\ -\int_0^a f(x)\,dx, & f \to \text{odd} \end{cases}$$

$$\therefore \int_{-a}^{a} f(x)\,dx = \begin{cases} \int_0^a f(x)\,dx + \int_0^a f(x)\,dx, & f \to \text{even} \\ -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx, & f \to \text{odd} \end{cases}$$

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & f \to \text{even} \\ 0, & f \to \text{odd} \end{cases}\left.\right)$$

Also $\displaystyle\int_{-\pi}^{\pi} \cos nx\,dx = \left.\frac{\sin nx}{n}\right|_{-\pi}^{\pi} = 0$

$$\int_{-\pi}^{\pi} \underbrace{\sin nx}_{\text{odd}}\,dx = \underline{\underline{0}}$$

$$\rightarrow \int_{-\pi}^{\pi} f(x)\,dx = a_0\pi \Rightarrow \boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

Again, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \int_{-\pi}^{\pi}\left(\frac{a_0}{2}\cos kx + \sum_{n=1}^{\infty} a_n \cos nx \cos kx + b_n \sin nx \cos kx\right)dx$$

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = 0 + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} a_n \cos nx \cos kx\,dx + 0$$

Consider the integral $I_1 = \int_{-\pi}^{\pi} \cos nx \cos kx\,dx$

$$I_1 = \frac{1}{2}\int_{-\pi}^{\pi}\left(\cos(n+k)x + \cos(n-k)x\right)dx$$

If $\underline{n=k}$ $\quad I_1 = \frac{1}{2}\int_{-\pi}^{\pi}\left(\cancel{\cos 2kx}^{\,0} + 1\right)dx$

$$= \frac{1}{2}(2\pi) = \pi$$

$n \neq k \Rightarrow I_1 = \frac{1}{2}\int_{-\pi}^{\pi}\left(\cos(n+k)x + \cos(n-k)x\right)dx$

$$= \frac{1}{2}\left[\frac{\sin(n+k)x}{n+k} + \frac{\sin(n-k)x}{n-k}\right]_{-\pi}^{\pi}$$

$$I_1 = 0$$

$$\therefore \int_{-\pi}^{\pi} \cos nx \cos kx\,dx = \begin{cases} 0, & n=k \\ \pi, & n\neq k \end{cases}$$

$$\therefore \int_{-\pi}^{\pi} f(x)\cos kx\,dx = \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} a_n \cos nx \cos kx\,dx$$

$$= \int_{-\pi}^{\pi} a_1 \overbrace{\cos x \cos kx}^{0}\,dx + \int_{-\pi}^{\pi} a_2 \overbrace{\cos 2x \cos kx}^{0}\,dx + \dots$$

$$+ \underbrace{\int_{-\pi}^{\pi} a_k \cos kx \cos kx\,dx}_{= a_k(\pi)} + \dots \rightarrow 0$$

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \pi a_k$$

$$\Rightarrow \boxed{a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx}$$

Again consider, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n \sin nx$

$$\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \underbrace{\int_{-\pi}^{\pi}\frac{a_0 \sin kx}{2}dx}_{odd} + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} \underbrace{a_n \cos nx \sin kx}_{odd}\,dx + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} b_n \sin nx \sin kx\,dx$$

$$\int_{-\pi}^{\pi} f(x)\sin kx\,dx = 0 + 0 + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} b_n \sin nx \sin kx\,dx$$

Consider the integral

$$\int_{-\pi}^{\pi} \sin nx \sin kx\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-k)x - \cos(n+k)x\,dx$$

If $n=k \Rightarrow \frac{1}{2}\int_{-\pi}^{\pi}(1-\cos 2kx)\,dx = \pi$

If $n\neq k \Rightarrow \frac{1}{2}\left[\frac{\sin(n-k)x}{n-k} - \frac{\sin(n+k)x}{n+k}\right]_{-\pi}^{\pi} = 0$

$$\therefore \int_{-\pi}^{\pi} \sin nx \sin kx\,dx = \begin{cases} 0, & n\neq k \\ \pi, & n=k \end{cases}$$

Now

$$\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} b_n \sin nx \sin kx\,dx$$

$$= \overset{0}{\cancel{\int_{-\pi}^{\pi} b_1 \sin x \sin kx\,dx}} + \overset{0}{\cancel{\int_{-\pi}^{\pi} b_2 \sin 2x \sin kx\,dx}}$$

$$+ \dots + \int_{-\pi}^{\pi} b_k \underbrace{\sin kx \sin kx\,dx}_{\pi} + \dots$$

$$\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \pi b_k$$

$$\Rightarrow \boxed{b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx}$$

Limitation :-

If the image has sharp edges or minute information we can't represent using sines and cosines

That means we have to change the basis to represent the data.

Wavelets :-

In wavelets we have different basis

But we don't know which basis to choose w.r.t our given data.

* we have experiment with different basis like square/Haar wavelet and so many others.

SVD :

using this technique we are able to find the basis to represent data:

$$X = U \Sigma V^T$$

Let $X_{3\times2} = U_{3\times2} \; \Sigma_{2\times3} \; V^T_{3\times2}$

$$X = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$U \in AA^T$

$UU^T = U^TU = I$

$VV^T = V^TV = I$

$X = U \Sigma V^T$

$XX^T = U\Sigma V^T (V \Sigma^T U^T)$

$= U \Sigma \Sigma^T U^T$

$XX^T = U \Lambda U^T$

Now $XX^T U = U \Lambda U^T U$

$XX^T U = U \Lambda$

$\therefore$ U is the eigen vector of XX^T

$||^{ly}$ V is the eigen vector of X^TX

Eg:

$$X = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}_{3\times2}$$

$$XX^T = \underset{3\times2}{\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}} \underset{2\times3}{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}} = \begin{bmatrix} 10 & 8 & 6 \\ 8 & 8 & 8 \\ 6 & 8 & 10 \end{bmatrix}$$

$|XX^T - \lambda I| = 0 \rightarrow$ To get eigen values

$$\Rightarrow \begin{vmatrix} 10-\lambda & 8 & 6 \\ 8 & 8-\lambda & 8 \\ 6 & 8 & 10-\lambda \end{vmatrix} = 0$$

$(10-\lambda)((8-\lambda)(10-\lambda) - 64)$

$-8(80 - 8\lambda - 48)$

$+6(64 - 48 + 6\lambda)$

lets take $x = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$xx^T = \underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$|xx^T - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)\{(1-\lambda)^2\} + 1(-1(1-\lambda)) + 1(-(1-\lambda)) = 0$$

$$= (1-\lambda)\{(2-\lambda)(1-\lambda) - 1 - 1\} = 0$$

$$= \boxed{\lambda = 1} \quad \cancel{2} - 2\lambda - \lambda + \lambda^2 - \cancel{2} = 0$$

$$\Rightarrow \lambda^2 = 3\lambda \qquad \lambda(\lambda - 3) = 0$$

$$\boxed{\lambda = 3} \qquad \boxed{\lambda = 0}$$

$\Rightarrow \underline{\lambda = 3}$

$$(xx^T - \lambda)u = \begin{bmatrix} -1 & -1 & 1 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0$$

$$\Rightarrow -u_1 - u_2 + u_3 = 0$$

$$-u_1 - 2u_2 = 0 \Rightarrow \boxed{u_1 = -2u_2}$$

$$\boxed{u_1 = 2u_3}$$

Let $\boxed{u_2 = -1}$

Then $\boxed{u_1 = 2}$, $\boxed{u_3 = 1}$

$$\vec{v}_1 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\hat{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

with $\underline{\lambda = 1}$

$$\begin{vmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 01 & 0 & 0 \end{vmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \Rightarrow \begin{array}{l} u_1 - u_2 + u_3 = 0 \\ -u_1 = 0 \Rightarrow \boxed{u_1 = 0} \\ \boxed{u_2 = u_3} \end{array}$$

Let $\boxed{u_2 = 1}$

$$\therefore \hat{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

with $\underline{\lambda = 0}$

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 2u_1 - u_2 + u_3 = 0$$

$$-u_1 + u_2 = 0 \Rightarrow \boxed{u_1 = u_2}$$

$$\boxed{u_1 = -u_3}$$

Let $\boxed{u_3 = -1} \Rightarrow \boxed{u_1 = 1} \Rightarrow \boxed{u_2 = 1}$

$$\hat{U_3} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$\lambda = 3$ $\lambda = 1$ $\lambda = 0$

$$U = \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{1} & 0 \\ 0 & 0 & \sqrt{0} \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{bmatrix}$$

$\lambda_1 > \lambda_2 > \lambda_3$

there $\lambda_1 = 3$ $\lambda_2 = 1$ $\lambda_3 = 0$

$X^T X$ - using this matrix we find $\underline{\underline{V}}$ matrix

To find the principal componets of X we multiply with U

$$XU = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

3×2 $\quad$ 3×3

2×3

we were unable to multiply

$\Rightarrow$ unable to find principal components.

Lets compute

$$X^T X = \underset{2 \times \boxed{3}}{\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}} \underset{\boxed{3} \times 2}{\begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Now $|X^T X - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)^2 = 1$$

$2 - \lambda = 1$ $\qquad$ $2 - \lambda = -1$

$\boxed{\lambda = 1}$ $\qquad$ $\boxed{\lambda = 3}$

<u>FOR $\lambda = 1$</u> Lets compute $\hat{v_1}$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_{11} + v_{12} = 0 \Rightarrow \boxed{v_{11} = -v_{12}}$$

Let $v_{12} = -1$

$$\boxed{\hat{v_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

For $\lambda = 3$ we get $\hat{v_2}$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \boxed{v_{21} = v_{22}}$ if $v_{22} = 1$ then:

$$\therefore \hat{v_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now $V = [\hat{v_2} \ \hat{v_1}]$ [$\because$ arranged in decreasing order of λ]

$$\boxed{V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}$$

Now lets try to find the principal components of X

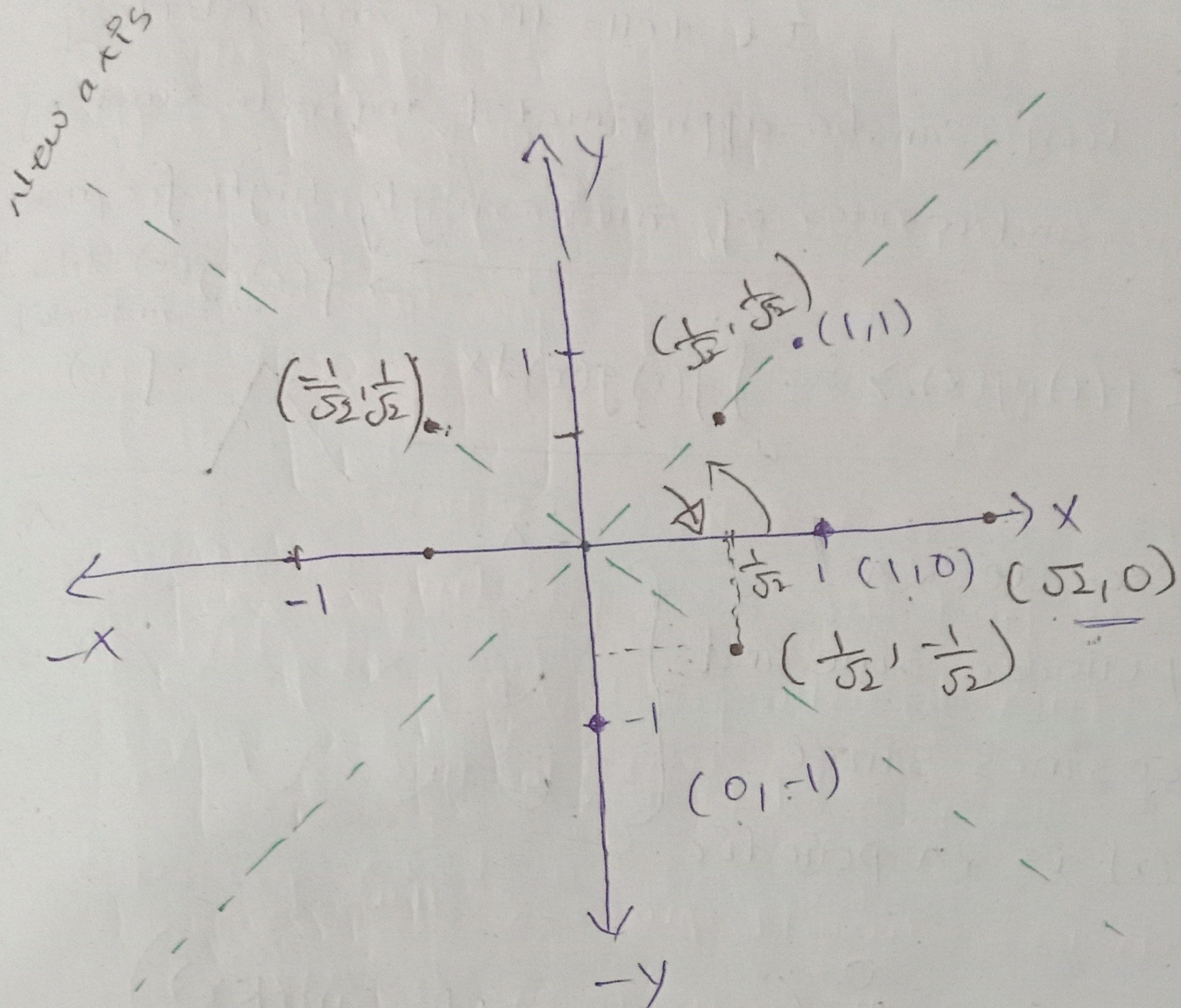
$$P = XV$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$3 \times \boxed{2 \ 2} \times 2$

$(P) =$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}$$



$x=0 \quad y=0$

$(1,1) \cancel{\longrightarrow} (\sqrt{2},0)$	$(1,1) \longrightarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
$(0,-1) \longrightarrow \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$(0,-1) \longrightarrow \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
$(1,0) \cancel{\longrightarrow} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$(1,0) \longrightarrow (\sqrt{2}, 0)$